k-core (bootstrap) percolation on complex networks: Critical phenomena and nonlocal effects

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(Received 5 March 2006; published 1 May 2006)

We develop the theory of the k-core (bootstrap) percolation on uncorrelated random networks with arbitrary degree distributions. We show that the k-core percolation is an unusual, hybrid phase transition with a jump emergence of the k-core as at a first order phase transition but also with a critical singularity as at a continuous transition. We describe the properties of the k-core, explain the meaning of the order parameter for the k-core percolation, and reveal the origin of the specific critical phenomena. We demonstrate that a so-called "corona" of the k-core plays a crucial role (corona is a subset of vertices in the k-core which have exactly k neighbors in the k-core). It turns out that the k-core percolation threshold is at the same time the percolation threshold of finite corona clusters. The mean separation of vertices in corona clusters plays the role of the correlation length and diverges at the critical point. We show that a random removal of even one vertex from the k-core may result in the collapse of a vast region of the k-core around the removed vertex. The mean size of this region diverges at the critical point. We find an exact mapping of the k-core percolation to a model of cooperative relaxation. This model undergoes critical relaxation with a divergent rate at some critical moment.

DOI: 10.1103/PhysRevE.73.056101 PACS number(s): 05.10.−a, 05.40.−a, 05.50.+q, 87.18.Sn

I. INTRODUCTION

Random damage may crucially change the structure and function of a network and may even completely destroy it. The description of the destruction of the complex network architectures due to damage is a challenging direction in the multidisciplinary science of networks. Remarkably, much attention was attracted to the hierarchical organization of various real-world networks (the Internet, the WWW, cellular networks, etc.) and extracting and indexing of their highly interconnected substructures — k-cores, cliques, and others [1–5]. The question is, how does random damage change and destroy these substructures, in particular, the k-cores?

The *k-core* of a network is its largest subgraph whose vertices have degree at least k. In other words, each of vertices in the k-core has at least k nearest neighbors within this subgraph. The k-core of a graph may be obtained in the following way (the k-core algorithm or "the pruning rule"). Remove from the graph all vertices of degree less than k. Some of the rest vertices may remain with less than k edges. Then prune these vertices, and so on until no further pruning is possible. The result, if it exists, is the k-core. The notion of the k-core [6,7] is a natural generalization of the giant connected component in the ordinary percolation [8–13]. The k-core percolation implies the breakdown of the giant k-core at a threshold concentration of vertices or edges removed at random. In physics, the k-core percolation (the k-core percolation) on a random Bethe lattice was introduced in Ref. [14] for describing some magnetic materials. The k-core percolation on a random Bethe lattice was used as a basic model of the quadrupolar ordering in solid $(o\text{-}H_2)_x(p\text{-}H_2)_{1-x}$ mixtures [15], the rigidity percolation [16], the jamming in sphere packing [17], glassy dynamics [18], etc. An exact threshold of the emergence of the k-core in some basic random networks was found in Refs. [19,20]. Recently, a general exact solution of the k-core problem for damaged and undamaged uncorrelated networks with arbitrary degree distributions was obtained in Ref. [5].

These investigations revealed that the k-core percolation is featured by an unusual phase transition which differs strongly from the ordinary percolation. The latter emerges through a continuous phase transition occurring at a critical concentration p_c (percolation threshold) of the vertex occupation probability p [8–13]. (A vertex is occupied with the probability p and is removed with the complementary probability $Q=1-p$.) At concentrations $p>p_c$ the giant connected component of a network occupies a finite fraction M of the total number of vertices N in the network. At $p\to p_c$, M tends to zero, $M\propto(p-p_c)^{\beta}$. The standard mean-field exponent $\beta=1$ takes place in networks with a rapidly decaying degree distribution. In networks with a scale-free degree distribution $q^{-\gamma}$, exponent β deviates from the mean-field value at $2<\gamma<4$ [12]. In these networks, $p_c=0$ at $\gamma\leqslant 3$.

The k-core percolation for $k\geqslant 3$ demonstrates another behavior. When $p\to p_c(k)$, the relative size M_k of the giant k-core tends to a finite value $M_k[p_c(k)]$, and at $p<p_c(k)$ the k-core is absent. Note that the critical concentration $p_c(k)$ depends on k. In this respect, the k-core percolation looks similar to a first-order phase transition with a jump of M_k at the critical point. However, for a first-order phase transition, one would expect that M_k is an analytical function of p at $p>p_c(k)$. Contrary to these expectations, M_k shows a singular behavior [5,14,17,21]: $M_k(p)-M_k[p_c(k)]\propto[p-p_c(k)]^{1/2}$. Recently, a similar phase transition was observed by numerical simulations of the random packing of frictionless particles (jamming transition) [22]. In the case of the ordinary percolation, the critical phenomena are related to the divergence of the mean size of finite clusters (finite connected components). But what is the origin of critical phenomena for the k-core percolation? First important steps in resolving this question have been made in Refs. [17,21] where the important role of a so-called "corona" of the k-core was noted. The *corona* is a subset of vertices in the k-core which have exactly k nearest neighbors in the k-core.

In the present paper we develop the qualitative and exact theories of the k-core percolation on complex networks. Our consideration is based on an exact solution of the k-core percolation on uncorrelated networks with arbitrary degree distributions. Specifically, we use the configuration model — the maximally random graphs with a given degree distribution [23]. In the large network limit, in any finite neighborhood of a vertex, these graphs have a treelike local structure, i.e., without loops, see, e.g., Refs. [24,25]. Note that in treelike networks, finite k-cores are absent, and so we discuss only the giant k-core.

In Sec. II we present a qualitative picture and demonstrate that the critical behavior at the k-core percolation threshold is related to the critical behavior of the corona of the k-core. In Sec. III we overview an exact solution describing the k-core organization. In Sec. IV we discuss the statistics of edges in a network with the k-core and the meaning of the order parameter for the k-core percolation. The critical behavior of the order parameter is considered in Sec. V. Using generating functions, in Sec. VI we show that the k-core percolation threshold is at the same time the percolation threshold for corona clusters. At this point the mean size of corona clusters diverges. The distribution of corona clusters over sizes is found in Sec. VII. Specific correlations between vertices in the k-core are discussed in Sec. VIII. It is demonstrated that the mean intervertex distance in corona clusters plays the role of the correlation length. In Sec. IX we derive exact equations which describe the evolution of the degree distribution in the k-core with increasing concentration of randomly removed vertices. It is demonstrated that a removal of even one vertex may result in a vast damage of the k-core. The "diameter" of the damaged region tends to infinity at the k-core percolation threshold. In Sec. X we propose an exact mapping of the k-core percolation to a model of cooperative relaxation. This model undergoes critical relaxation with a divergent rate at some critical moment of the evolution.

II. RANDOM DAMAGING THE k CORE

In this section we qualitatively describe the k-core percolation in an uncorrelated random network for $k \geqslant 3$. We assume that a vertex in the original network is occupied with a probability p. Let M_k be the probability that a randomly chosen vertex in a given network belongs to the k-core. The k-core in itself is a random network which consists of vertices with degrees at least k. Corona vertices, i.e., vertices with the degree k, are distributed randomly in the k-core. They occupy a finite part of the k-core. We shall show in Sec. VI that at $p > p_c(k)$, the corona consists of only finite clusters, so that its relative size in the k-core is sufficiently small. Any vertex in the k-core may have a link with a corona vertex belonging to a finite corona cluster.

Let us study the change of the k-core size when removing vertices at random from the original network. Let the occupation probability p be diminished by a value Δp. This corresponds to a random removal of $N\Delta p$ vertices. We denote the corresponding decrease in the size of the k-core by

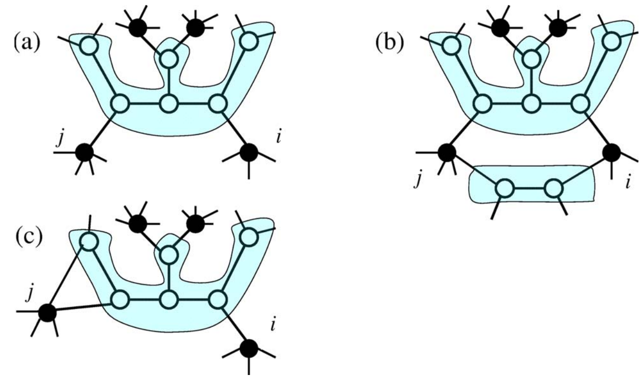

FIG. 1. (Color online) (a) A part of the $k=3$-core with a finite corona cluster which consists of vertices with exactly three edges. This cluster is shown as a grey region. Corona vertices are represented by open dots. In a treelike network only one corona cluster may connect two vertices, for example, vertices i and j on this figure. Removal of vertex i results in pruning all vertices which belong to the corona cluster. As a result, vertex j loses one neighbor (this is the neighboring corona vertex). The degree of vertex j decreases by 1. (b) In a network with loops, two or more corona clusters may connect together a pair of vertices in the k-core. (c) In networks with nonzero clustering, a vertex in the k-core may be attached to a corona cluster by two or more edges. In the cases (b) and (c) a removal of the vertex i results in pruning the corona vertices and, in turn, the degree of the vertex j is decreased by 2.

$N\Delta M_k = NM_k(p) - NM_k(p-\Delta p)$. This change is a quantity to be found. Firstly, there is a trivial contribution to $N\Delta M_k$ due to the removal of the deleted vertex from the k-core:

$$N\delta M_k \equiv N\Delta p\, \partial M_k/\partial p = \Delta p N M_k/p. \tag{1}$$

This can be seen from Eqs. (5) and (6).

Secondly, after removing a vertex together with its edges from the k-core we must prune all other vertices which will occur with degrees less than k. In fact, the removal of a single vertex i from the k-core results in the removal of the entire corona clusters attached to vertex i. Note that several corona clusters may be attached to a vertex with degree $n > k$ in the k-core. The removal of the corona clusters happens due to "the domino principle." Indeed, after removing vertex i, its nearest neighboring corona vertex loses one link with the k-core. This vertex must be pruned from the k-core, because now it has only $k-1$ links with the k-core. Due to this removal, each of second nearest neighbors of vertex i in the corona clusters also loses one link with the k-core and also must be pruned, and so on until all vertices in the corona clusters will be pruned one by one. This process is explained in Fig. 1(a), where a part of the $k=3$-core with a corona cluster is represented. Let $N_{\rm crn}$ be the mean total size of all corona clusters attached to a vertex in the k-core [26]. Then the second contribution to $N\Delta M_k$ is $N\delta M_k N_{\rm crn}$ which is the number of the deleted vertices in the k-core multiplied by $N_{\rm crn}$.

What happens with other vertices remaining in the k-core after the removal of vertex i together with corona clusters attached to i? If there is no loops, all nearest neighbors of the deleted vertex i and of the pruned corona vertices remain in the k-core. Their degrees decrease by 1 since each of these vertices loses one link with the k-core: $n \to n-1 \geqslant k$. On the other hand, in networks with loops, due to the pruning, a vertex may lose more than one connection to the k-core. Such situations are represented in Figs. 1(b) and 1(c).

Thus, in a treelike network,

$$N\Delta M_k = N\delta M_k + N\delta M_k N_{\text{crn}}. \tag{2}$$

In a differential form this equation looks as follows:

$$\frac{d \ln M_k}{d \ln p} = 1 + N_{\text{crn}}. \tag{3}$$

We will show in Sec. VI that corona clusters percolate exactly at the k-core percolation threshold $p_c(k)$ and that N_{crn} diverges as $[p-p_c(k)]^{-1/2}$. Consequently, according to Eq. (3), M_k demonstrates a critical singularity

$$M_k(p) - M_k[p_c(k)] \propto [p - p_c(k)]^{1/2}. \tag{4}$$

In Sec. VI we will also show that Eq. (3) is exact for uncorrelated networks with an arbitrary degree distribution (the configuration model) in the limit $N \to \infty$.

III. BASIC EQUATIONS

In this section we develop an exact formalism for calculating various k-core's characteristics. This approach is based on our paper [5].

We consider an uncorrelated network with a given degree distribution $P(q)$ — the so-called configuration model. We assume that a vertex in the network is occupied with the probability p. In this treelike network, the giant k-core coincides with the infinite $(k-1)$-ary subtree. By definition, the m-ary tree is a tree, where all vertices have branching at least m. The introduction of the $(k-1)$-ary tree notion allows one to strictly define the order parameter in the k-core problem for treelike networks (see below).

Let R be the probability that a given end of an edge of a network is not the root of an infinite $(k-1)$-ary subtree. Then, the probability $M_k(n)$ that a vertex chosen at random has exactly $n \geqslant k$ neighbors in the k-core is given by the equation

$$M_k(n) = p \sum_{q \geqslant n} P(q) C_n^q R^{q-n} (1-R)^n. \tag{5}$$

Here, $P(q)$ is the probability that a randomly chosen vertex in the original undamaged network has degree q. $C_n^m R^{q-n}(1-R)^n$ is the probability that a vertex with degree q has $q-n$ neighbors which are not roots of infinite $(k-1)$-ary subtrees and n neighbors which are roots of infinite $(k-1)$-ary subtrees. The combinatorial multiplier $C_n^m = m!/(m-n)!n!$ gives the number of ways one can choose n neighbors from q neighbors. A vertex belongs to the k-core if at least k its neighbors are roots of infinite $(k-1)$-ary subtrees. So the probability M_k that a vertex belongs to the k-core is equal to

$$M_k = \sum_{n \geqslant k} M_k(n). \tag{6}$$

Schematic Fig. 2 explains Eqs. (5) and (6). Note that for the ordinary percolation we must set $k=1$ in this equation. The

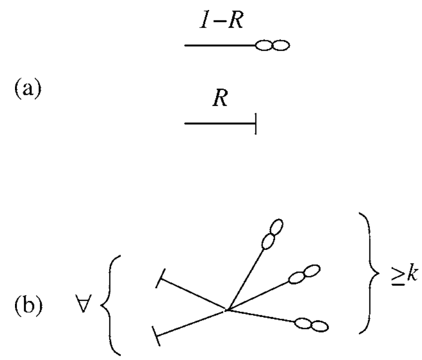

FIG. 2. (a) Schematic representation of the order parameter. $1-R$ is the probability that a given end of a randomly chosen edge in an undamaged network is a root of an infinite $(k-1)$-ary subtree. R is the probability that a given end of an edge is not a root of an infinite $(k-1)$-ary subtree. (b) Schematic view of vertex configurations contributing to M_k which is the probability that a vertex is in the k-core [see Eqs. (5) and (6)]. A vertex in a treelike network belongs to the k-core if at least k its nearest neighbors are the roots of infinite $(k-1)$-ary subtrees. The symbol ∀ shows that the number of nearest neighbors which are not roots of infinite $(k-1)$-ary subtrees is arbitrary.

number of vertices in the k-core is equal to NM_k.

The probability R plays the role of the order parameter in our problem. Due to using the $(k-1)$-ary trees, R is defined in such a way that it is independent of the second end of the edge — whether it belongs or does not belong to the k-core.

An end of an edge is not a root of an infinite $(k-1)$-ary subtree if at most $k-2$ its children branches are roots of infinite $(k-1)$-ary subtrees. This leads to the following exact equation for R [5]:

$$R = 1 - p + p \sum_{n=0}^{k-2} \left[\sum_{i=n}^{\infty} \frac{(i+1)P(i+1)}{z_1} C_n^i R^{i-n} (1-R)^n \right]. \tag{7}$$

Let us explain this equation. (i) The first term, $1-p \equiv Q$, is the probability that the end of the edge is unoccupied. (ii) $C_n^i R^{i-n}(1-R)^n$ is the probability that if a given end of the edge has i children (i.e., other edges than the starting edge), then exactly n of them are roots of infinite $(k-1)$-ary subtrees. $(i+1)P(i+1)/z_1$ is the probability that a randomly chosen edge leads to a vertex with branching i. $z_1 = \Sigma_q qP(q)$ is the mean number of the nearest neighbors of a vertex in the graph. Thus, in the square brackets, we present the probability that a given end of the edge has exactly n edges, which are roots of infinite $(k-1)$-ary subtrees. (iii) Finally, we take into account that n must be at most $k-2$. In an alternative form, Eq. (7) may be written as follows;

$$1 - R = p \sum_{n=k-1}^{\infty} \left[\sum_{i=n}^{\infty} \frac{(i+1)P(i+1)}{z_1} C_n^i R^{i-n} (1-R)^n \right]. \tag{8}$$

This equation shows that a given end of an edge is a root of an infinite $k-1$-ary tree with the probability $1-R$ if it has at least $k-1$ children (we sum over $n \geqslant k-1$) which also are

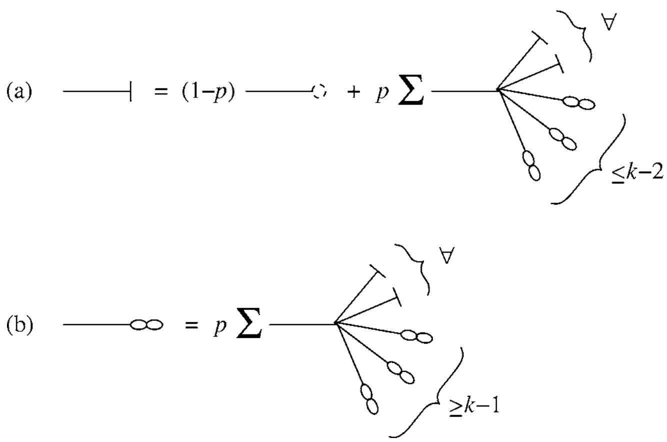

FIG. 3. (a) and (b) are graphic representations of Eqs. (7) and (8), respectively. In (a) the open circle with a dashed boundary represents an unoccupied vertex. Other notations are explained in the caption to Fig. 2.

roots of an infinite $(k-1)$-ary subtree. Equations (7) and (8) are graphically represented in Fig. 3.

Introducing a function

$$\Phi_k(R) = \sum_{n=0}^{k-2} \sum_{i=n}^{\infty} \frac{(i+1)P(i+1)}{z_1} C_n^i R^{i-n}(1-R)^n, \qquad (9)$$

we rewrite Eq. (7) in a concise form:

$$R = 1 - p + p\Phi_k(R). \qquad (10)$$

If this equation has only the trivial solution $R=1$, there is no giant k-core. The emergence of a nontrivial solution corresponds to the birth of the giant k-core. The k-core is described by the lowest nontrivial solution $R<1$.

The structure of the k-core is essentially determined by its degree distribution $P_k(n)$:

$$P_k(n) \equiv \frac{M_k(n)}{M_k}. \qquad (11)$$

$P_k(n)$ is the probability to find a vertex with degree n in the k-core. Note that the k-core of an uncorrelated network in itself is an uncorrelated graph, and so it is completely described by its degree distribution $P_k(n)$ [27]. We will extensively use this circumstance. The corona occupies a fraction $P_k(k)$ of the k-core. Therefore, the total number of vertices in the corona is equal to $NM_kP_k(k)$. The mean degree of vertices in the k-core is

$$z_{1k} = \sum_{n\geqslant k} P_k(n)n. \qquad (12)$$

Comparing Eqs. (12) and (8), we get an important relationship between z_{1k}, M_k, and $1-R$:

$$z_{1k}M_k = z_1(1-R)^2. \qquad (13)$$

Below, in Sec. IV, we will discuss its meaning.

In the general case the analytical solution of Eq. (7) is unknown. But it can be obtained numerically. By using this solution, one can calculate all basic characteristics of the k-core structure of an arbitrary uncorrelated network [5].

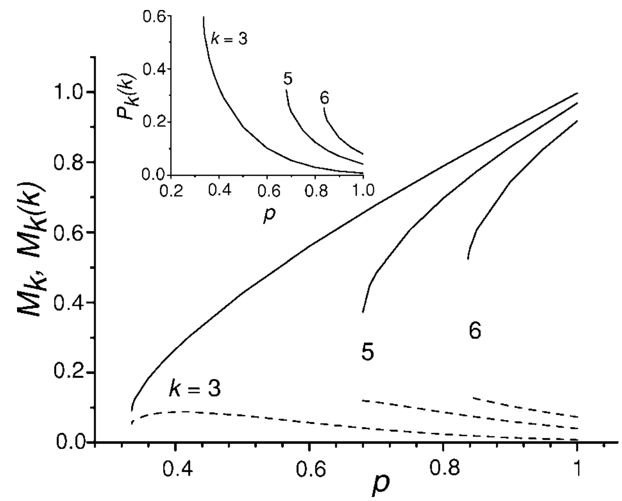

FIG. 4. Dependence of the sizes of the k-core and the corona, M_k and $M_k(k)$, respectively, on the occupation probability p in the Erdős-Rényi graphs with the mean degree $z_1=10$. Solid lines show M_k versus p, and dashed lines show $M_k(k)$ versus p at $k=3$, 5, and 6. Notice that both M_k and $M_k(k)$ have a square root singularity at the k-core percolation thresholds, but in the curves $M_{k=5,6}(k)$ this singular addition is practically unnoticeable. Notice the nonmonotonous dependence $M_k(k)$. The inset shows the fraction $P_k(k)$ of the corona vertices in the k-core versus p.

Some results of the numerical solution of Eq. (7) for the Erdős-Rényi graph with $z_1=10$ are represented in Fig. 4. More results may be found in Ref. [5]. This figure displays the dependences of the sizes of the k-core, the corona and the fraction of the corona vertices in the k-core on the occupation probability p for several values of k. One can see that far from the critical point $p_c(k)$ the size of the corona is small in comparison to the size of the k-core. However, close to $p_c(k)$ the corona occupies a noticeable fraction of the k-core.

Let us consider a network with a scale-free degree distribution $P(q) \propto q^{-\gamma}$. The case $2<\gamma\leqslant 3$ is realized in most important real-world networks. With γ in this range, the mean number of the nearest neighbors of a vertex in a network z_2 diverges if $N\to\infty$. Solving analytically Eq. (10), we find that the size of the k-core decreases with increasing k as follows [5]:

$$M_k = p^{2/(3-\gamma)}(q_0/k)^{(\gamma-1)/(3-\gamma)}, \qquad (14)$$

where q_0 is the minimum degree in the initial (undamaged) network. Vertices which belong to the k-core, but do not belong to the $(k+1)$-core, form the k-shell of the size $S_k = M_k - M_{k+1}$. Using Eq. (14), at $k \gg 1$ we find

$$S_k \propto (q_0/k)^{2/(3-\gamma)}. \qquad (15)$$

The asymptotic behavior given by Eqs. (14) and (15) agrees well with an empirical analysis of the k-core architecture of the Internet on the Autonomous Systems level [1,2].

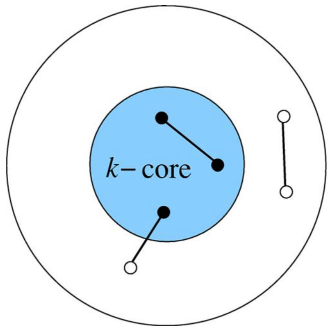

FIG. 5. (Color online) Schematic representation of the three types of edges in a network (large circle) with the k-core (grey central region): (i) edges between vertices in the k-core (links between two black dots), (ii) edges between vertices which do not belong to the k-core (links between two open dots), and (iii) edges between vertices in the k-core and vertices which do not belong to the k-core (links between black and open dots).

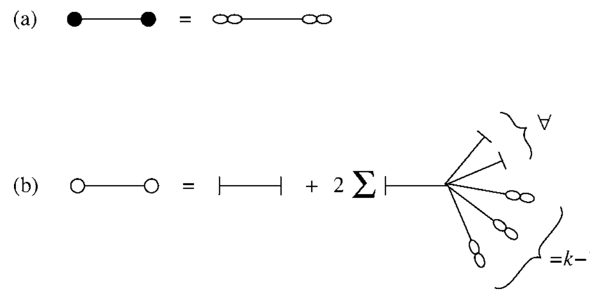

FIG. 6. Schematic representations of the probabilities that an edge connects together vertices of the k-core or that it connects vertices outside the k-core, (a) and (b), respectively.

IV. STATISTICS OF EDGES AND THE ORDER PARAMETER

Let us consider edges in an uncorrelated network with the k-core. We start with the case $p=1$. There are three types of edges: (i) edges which connect two vertices in the k-core, (ii) edges connecting two vertices which do not belong to the k-core, and (iii) edges connecting together a vertex in the k-core and the other one which do not belong to the k-core. These types of connections in a network are schematically shown in Fig. 5. Let L_k, L_0, and L_{0k} be the total numbers of edges of these three types in the network, respectively. The sum of these numbers gives the total number $L=Nz_1/2$ of edges in the initial network

$$L_0 + L_k + L_{0k} = L. \tag{16}$$

The ratios L_k/L, L_0/L, and L_{0k}/L are probabilities that a randomly chosen edge in the initial network is of the type (i), (ii), or (iii), respectively. Because $L_k=Nz_{1k}M_k/2$, we can rewrite Eq. (13) in the form

$$\frac{L_k}{L} = (1-R)^2. \tag{17}$$

This equation has a simple explanation. It shows that the probability to find an edge which connects two vertices in the k-core is equal to the probability that both its ends are roots of the $(k-1)$-ary tree, that is, $(1-R)^2$ [see Fig. 6(a)]. On the other hand, Eq. (17) explains the meaning of the order parameter R via the relationship with the measurable parameters: $1-R=\sqrt{L_k/L}$.

One should note that Eq. (17) is also valid at $p<1$ since it follows from the exact equation (13). In this case, L_k must be replaced by the number $L_k(p)$ of edges in a damaged k-core while L remains the total number of edges in the initial network.

Let us find the probability L_0/L that an edge chosen at random in the network connects two vertices which do not belong to the k-core. We stress that L_0/L is not equal to R^2 as one could naively expect, but is larger, see Fig. 6(b). Indeed, in addition to configurations where both the ends of an edge are not the roots of infinite $(k-1)$-ary trees — the R^2 contribution — one must take into account extra configurations. In these additional configurations, one end of the edge is not the root of an infinite $(k-1)$-ary tree, but the second end has exactly $k-1$ childrens which are roots of infinite $(k-1)$-ary trees. This second vertex does not belong to the k-core as it should be. Thus we have

$$L_0/L = R^2 + 2R\sum_{q\geqslant k}\frac{qP(q)}{z_1}C_{k-1}^{q-1}R^{q-k}(1-R)^{k-1}. \tag{18}$$

Comparing the sum in Eq. (18) and the probability $M_k(k)$ given by Eq. (5) at $n=k$, we get

$$L_0/L = R^2 + 2R\frac{kM_k(k)}{z_1(1-R)}. \tag{19}$$

Equations (16), (17), and (19) establish nontrivial relationships between independently measurable network parameters L, L_k, L_0, L_{0k}, and $M_k(k)$. These relations may be used as a criterion of the validity of the tree ansatz for various networks.

Let us now touch upon the case $p<1$. After random removal vertices from an uncorrelated network, we again get an uncorrelated network. Therefore, at $p<1$, we may still use the same formulas (17)–(19) but with substituted characteristics of the damaged network — the number of edges, the mean degree, etc. [28].

V. k-CORE PERCOLATION THRESHOLD

When decreasing the occupation probability p, the k-core decreases in size and disappears at a critical concentration $p_c(k)$. According to Ref. [5], the critical concentration $p_c(k)$ is determined by the following equation:

$$p_c(k)\Phi_k'(R_c) = 1. \tag{20}$$

Here, R_c is a critical value of the order parameter R at the birth point of the k-core. At $p<p_c(k)$, Eq. (10) has only the trivial solution $R=1$, and the giant k-core does not exist. The derivative $\Phi_k'(R)\equiv d\Phi_k(R)/dR$ is determined by the following equation:

$$p\Phi_k'(R) = p\sum_{q\geqslant k}\frac{qP(q)}{z_1}C_{k-2}^{q-1}(q+1-k)R^{q-k}(1-R)^{k-2}$$
$$= k(k-1)P_k(k)/z_{1k}. \quad (21)$$

Using this equation, the condition (20) for the k-core percolation threshold may be rewritten in the form

$$k(k-1)P_k(k)/z_{1k} = 1. \quad (22)$$

Let us consider the behavior of R near the phase transition in an uncorrelated complex network with a finite mean number z_2 of the second neighbors of a vertex, $z_2=\Sigma_q q(q-1)P(q)$. At p near $p_c(k)$, i.e., at $0 < p - p_c(k) \ll 1$, Eq. (7) has a nontrivial solution

$$R_c - R \propto [p - p_c(k)]^{1/2}. \quad (23)$$

This demonstrates that R has a jump at $p= p_c(k)$ [from $R=R_c$ to $R=1$] as at an ordinary first-order phase transition and a singular behavior as at a continuous phase transition [14]. The derivative dR/dp diverges,

$$\frac{dR}{dp} = -\frac{1-R}{p[1-p\Phi_k'(R)]} \propto -[p-p_c(k)]^{-1/2}, \quad (24)$$

since at $p\to p_c(k)+0$ we have

$$1 - p\Phi_k'(R) \propto [p - p_c(k)]^{1/2}. \quad (25)$$

This singularity suggests intriguing critical phenomena near the threshold of the k-core percolation.

In contrast, in networks with infinite z_2, instead of the hybrid phase transition, the k-core percolation becomes an infinite order phase transition [5], similarly to the ordinary percolation in this situation [9]. In this case the entire k-core organization of a network is extremely robust against random damage.

VI. GENERATING FUNCTIONS FOR CORONA CLUSTERS

Using the approach of Refs. [9,10], we introduce the generating function $H_1(x)$ of the probability that an end of a randomly chosen edge in the k-core belongs to a finite corona cluster of a given size

$$H_{1k}(x) = 1 - \frac{kP_k(k)}{z_{1k}} + x\frac{kP_k(k)}{z_{1k}}[H_{1k}(x)]^{k-1}. \quad (26)$$

Here $kP_k(k)/z_{1k}$ is the probability that an end of an edge chosen at random in the k-core belongs to the corona. In turn, $1-kP_k(k)/z_{1k}$ is the complementary probability that the end of the edge does not belong to the corona. We have $H_{1k}(1)=1$.

We introduce the generating function $H_{0k}(x)$ for the size of a corona cluster attached to a vertex in the k-core:

$$H_{0k}(x) = \sum_q P_k(q)[H_{1k}(x)]^q. \quad (27)$$

Using this function, one can calculate the mean total size N_{crn} of the corona clusters attached to a vertex randomly chosen in the k-core:

$$N_{\mathrm{crn}} = \left.\frac{dH_{0k}(x)}{dx}\right|_{x=1}. \quad (28)$$

Differentiating Eqs. (26) and (27) over x gives

$$N_{\mathrm{crn}} = \frac{kP_k(k)}{1-k(k-1)P_k(k)z_{1k}^{-1}}. \quad (29)$$

Inserting Eqs. (21) and (25) into Eq. (29), we find that N_{crn} diverges at the critical point

$$N_{\mathrm{crn}} \propto [p - p_c(k)]^{-1/2}. \quad (30)$$

This means that at $p=p_c(k)$ the corona is in its "percolation transition threshold." Note, however, that the k-core and its corona are absent at $p < p_c(k)$, so that there is no giant connected corona above this threshold, in contrast to the ordinary percolation. Equation (22) resembles the condition $pz_2/z_1=1$ of the emergence of the giant connected component in the ordinary percolation.

The exact derivation of Eq. (3) is based on the following steps. We differentiate Eq. (6) for M_k over p:

$$\frac{dM_k}{dp} = \frac{M_k}{p} - kP_k(k)\frac{M_k}{(1-R)}\frac{dR}{dp}. \quad (31)$$

Inserting Eqs. (21), (24), and (29) into Eq. (31), we get Eq. (3).

VII. SIZE DISTRIBUTION OF CORONA CLUSTERS

Let $\mathfrak{N}_{\mathrm{crn}}(s)$ be the number of corona clusters of size s in the k-core. Because the total number of vertices in the corona clusters is equal to $NM_kP_k(k)$, we obtain the following condition:

$$\sum_{s=1}^{s_{\max}} s\mathfrak{N}_{\mathrm{crn}}(s) = NM_kP_k(k), \quad (32)$$

where $s_{\max}$ is the size of the largest corona cluster. We introduce a function

$$\Pi_k(s) \equiv \frac{s\mathfrak{N}_{\mathrm{crn}}(s)}{NM_kP_k(k)}, \quad (33)$$

which is the probability that a randomly chosen corona vertex belongs to a corona cluster of size s. The function $\Pi_k(s)$ is related to the generating function $G_{\mathrm{crn}}(x)$:

$$\Pi_k(s) = \frac{1}{s!}\left.\frac{d^sG_{\mathrm{crn}}(x)}{dx^s}\right|_{x=0}, \quad (34)$$

where $G_{\mathrm{crn}}(x)=x[H_{1k}(x)]^k$. There is a simple relationship between N_{crn} and the mean size s_{crn} of a corona cluster to which a randomly chosen corona vertex belongs:

$$s_{\text{crn}} \equiv \sum_{s=1}^{s_{\max}} s\Pi_k(s) = 1 + \frac{k}{z_{1k}} N_{\text{crn}}. \tag{35}$$

At $s \gg 1$ the probability $\Pi_k(s)$ has the usual asymptotic form [10]

$$\Pi_k(s) \approx C s^{-\alpha} e^{-s/s^*}. \tag{36}$$

Here C is a constant. Exponent α and the parameter $s^* = 1/\ln|x^*|$ are determined by the type and the position x^* of the singularity of the function $H_{0k}(x)$, nearest to $x=1$. Solving Eq. (26) and inserting the obtained solution into Eq. (27), we find that at $p \to p_c(k)+0$ the generating functions $H_{1k}(x)$ and $G_{\text{crn}}(x)$ have a square-root singularity

$$G_{\text{crn}}(x) \propto H_{1k}(x) \propto (1-x)^{1/2}. \tag{37}$$

In the case $k=3$, Eq. (26) is solved exactly,

$$H_{1k=3}(x) = \frac{1-(1-x/x^*)^{1/2}}{2ax}, \tag{38}$$

where $a = 3P_3(3)/z_{1,k=3}$ and $x^* = 1/[4a(1-a)]$. At the critical point, Eq. (22), we have $2a=1$ and, therefore, $x^*=1$. At p near $p_c(k)$, the parameter s^* diverges,

$$s^* \approx 1/(1-2a)^2 \propto 1/[p-p_c(k)]. \tag{39}$$

The singularity (37) results in the standard mean-field exponent $\alpha=3/2$. At the critical point $p=p_c(k)$, the distribution function is

$$\Pi_k(s) \propto s^{-3/2}. \tag{40}$$

It gives $\mathfrak{N}_{\text{crn}}(s) \propto \Pi_k(s)/s \sim s^{-5/2}$. In scale-free networks with a degree distribution $P(q) \sim q^{-\gamma}$, this is valid for any $\gamma > 3$. In contrast, in the ordinary percolation on scale-free uncorrelated networks, exponent α differs from the standard mean-field value $3/2$ if $2 < \gamma < 4$ [12,29].

Let us estimate the size $s_{\max}$ of the largest corona cluster. We use the condition that there is only one corona cluster with the size $s \geq s_{\max}$:

$$\int_{s_{\max}}^{\infty} \mathfrak{N}_{\text{crn}}(s) ds = N M_k P_k(k) \int_{s_{\max}}^{\infty} \Pi_k(s) s^{-1} ds = 1. \tag{41}$$

Using asymptotics (36), at $p > p_c(k)$ we obtain

$$s_{\max} \propto \ln N. \tag{42}$$

At the critical point $p=p_c(k)$, using the distribution function $\Pi_k(s) \propto s^{-3/2}$, we get

$$s_{\max} \propto N^{2/3}, \tag{43}$$

$$s_{\text{crn}} \propto \sqrt{s_{\max}} \propto N^{1/3}. \tag{44}$$

Equation (43) coincides with a result for the maximum size of a connected component at the birth point of the giant connected component in classical random graphs [30] and in uncorrelated networks where three first moments of a degree distribution converge [29]. However, relation (43) essentially differs from that for the maximum size of a connected component if the third moment of the degree distribution diverges in the infinite network limit.

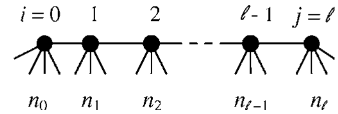

FIG. 7. Diagrammatic representation of the mean number $\mathcal{P}_l(n_0, n_1, \ldots, n_l)$ of ways to reach a vertex which is at distance l from a vertex $i=0$ in the k-core. The path goes through vertices $m=1,2,\ldots,l-1$ with degrees n_m in the k-core.

VIII. THE CORRELATION LENGTH

In this section we consider correlations between vertices in the k-core with $k \geq 3$. Let us chose at random a vertex i in the k-core. We aim to find the mean number $\mathcal{P}_l$ of vertices in the k-core which may be reached from i following a path of a length l. In the configuration model the giant k-core is a unique and simply connected subgraph. Therefore, all $l-1$ vertices on a path connecting i and j must also belong to the k-core. $\mathcal{P}_l$ is given by the following relation:

$$\mathcal{P}_l(n_0, n_1, \ldots, n_l) = P_k(n_0) n_0 \prod_{m=1}^{l-1} \left[\frac{P_k(n_m) n_m (n_m - 1)}{z_{1k}} \right] \frac{P_k(n_l) n_l}{z_{1k}}. \tag{45}$$

A diagrammatic representation of $\mathcal{P}_l$ is shown in Fig. 7. Here n_m is the degree of vertex m, where $m=0,1,\ldots,l$, on a path connecting i and j in the k-core. For the sake of convenience we set $i \equiv 0$ and $j \equiv l$. Here $P_k(n_0)$ is the probability that i has the degree n_0 in the k-core. The multiplier n_0 gives the number of ways to reach vertex 1 from $i=0$ following along any of its n_0 edges. In the brackets, $P_k(n_m) n_m / z_{1k}$ is the probability that an edge outgoing from a vertex $m-1$ leads to a vertex m of degree n_m. The multiplier $n_m - 1$ gives the number of ways to leave this vertex. Finally, $P_k(n_l) n_l / z_{1k}$ is the probability that the final edge on the path leads to the destination vertex l of degree n_l.

Now it is easy to find the number $N_{\text{crn}}(l)$ of corona vertices which are at a distance l from a randomly chosen vertex in the k-core and belong to corona clusters attached to this vertex. We set $n_1 = n_2 = \cdots = n_l = k$ and sum over degree n_0 of the starting vertex 0 in Eq. (45). Using Eq. (12) gives

$$N_{\text{crn}}(l) = \sum_{n_0 \geq k} \mathcal{P}_l(n_0, n_1 = k, \ldots, n_l = k) = k P_k(k) e^{-(l-1)/\lambda}. \tag{46}$$

Here we have introduced the correlation length

$$\lambda \equiv -\frac{1}{\ln[p\Phi_k'(R)]} = -\frac{1}{\ln[k(k-1)P_k(k)/z_{1k}]}. \tag{47}$$

In accordance with Eq. (25), at $p \to p_c(k)+0$ the parameter λ diverges

$$\lambda \propto [p - p_c(k)]^{-1/2}. \tag{48}$$

Summing $N_{\text{crn}}(l)$ over l, we reproduce Eq. (29) for N_{crn}

$$N_{\mathrm{crn}} = \sum_{l=1}^{\infty} N_{\mathrm{crn}}(l) = \frac{kP_k(k)}{1 - \exp[-1/\lambda]}. \tag{49}$$

Let us determine the mean intervertex distance $r_{\mathrm{crn}}(k)$ between vertices in corona clusters. We use the quantity $\mathcal{P}_l(n_0, n_1, \ldots, n_l)$ and set $n_0 = n_1 = \cdots = n_l = k$. We find

$$r_{\mathrm{crn}}(k) \equiv \frac{\sum_{l=1}^{\infty} l\mathcal{P}_l(k,k,\ldots,k)}{\sum_{l=1}^{\infty} \mathcal{P}_l(k,k,\ldots,k)} = \frac{1}{1 - \exp[-1/\lambda]}. \tag{50}$$

At p close to $p_c(k)$, the correlation length $\lambda \gg 1$ and therefore $r_{\mathrm{crn}}(k) \approx \lambda$.

IX. NONLOCAL EFFECTS IN THE k-CORE PERCOLATION

In Sec. II we have shown that a removal of a vertex from the k-core leads to pruning corona clusters attached to the vertex. In this section we will demonstrate that a removal of even one vertex from the k-core, $k \geq 3$, influences degrees of vertices in a vast region of the k-core around this vertex. Moreover, the size of this damaged region diverges at the critical point.

Let $N\Delta p$ vertices be removed at random from the initial network. As a result, the total number of vertices with degree n in the k-core is changed,

$$N\Delta M_k(n) = NM_k(p,n) - NM_k(p - \Delta p, n). \tag{51}$$

Let us find $N\Delta M_k(n)$. With probability $M_k(n)$, a removed vertex may have degree n in the k-core. Therefore, there is a trivial contribution to $N\Delta M_k(n)$:

$$N\delta M_k(n) = N\Delta p\, \partial M_k(n)/\partial p = N\Delta p M_k(n)/p. \tag{52}$$

Removal of a vertex i in the k-core may influence on the degree of a vertex j which is at a distance l from i. If j is a nearest neighbor of i, then the degree n of vertex j will be decreased by 1. If $l > 1$, then the probability of this effect is determined by the probability that j and i are connected by a chain of corona vertices. If vertex i is removed, then all vertices of a corona cluster attached to i also must be pruned from the k-core due to the domino principle. As a result, vertex j loses one neighbor in the k-core. Let $V(l,n)$ be the mean number of vertices of degree n which are connected to a randomly chosen vertex i in the k-core by a chain of corona vertices of length l. Removal at random $N\delta M_k$ vertices results in a decrease of $M_k(n)$ by a quantity

$$N\delta M^{(2)}(n) = N\delta M_k \sum_{l=1}^{\infty} V(l,n). \tag{53}$$

At the same time vertices with degree $n+1$ may also lose one edge with the k-core. After the pruning, they have n edges within the k-core. This effect increases $M_k(n)$ by a quantity

$$N\delta M^{(3)}(n) = -N\delta M_k \sum_{l=1}^{\infty} V(l,n+1). \tag{54}$$

Note that only in networks with loops, vertices in the k-core may change their degree by 2 during the pruning. Thus, in a treelike network there are only three contributions to $N\Delta M_k(n)$:

$$N\Delta M_k(n) = N\delta M_k(n) + N\delta M_k \sum_{l=1}^{\infty} V(l,n) - N\delta M_k \sum_{l=1}^{\infty} V(l,n+1). \tag{55}$$

$V(l,n)$ is given by the probability $\mathcal{P}_l$, Eq. (45):

$$V(l,n) = \sum_{n_0 \geq k} \mathcal{P}_l(n_0, n_1 = k, \ldots, n_{l-1} = k, n_l = n) = nP_k(n)e^{-(l-1)/\lambda}. \tag{56}$$

Inserting this result into Eq. (55), in the limit $\Delta p \to 0$, we get the main result of the present section:

$$\frac{dM_k(n)}{d\ln p} = M_k(n) + rnM_k(n) - r(n+1)M_k(n+1), \tag{57}$$

where

$$r = \frac{1}{1 - \exp[-1/\lambda]} = \left[1 - \frac{k(k-1)M_k(k)}{\sum_{n=k}^{q_{\mathrm{cut}}} nM_k(n)}\right]^{-1}. \tag{58}$$

The parameter r determines the mean size of a region in the k-core which is damaged by a removal of one vertex chosen at random. One should stress that r depends on the entire degree distribution in the k-core: $r = r\{M_k(n)\}$. At p close to $p_c(k)$, we have $r \propto \lambda$. Therefore, at $p \to p_c(k)$, this size diverges. Interestingly, the parameter r is equal to the mean intervertex distance r_{crn} in corona clusters given by expression (50).

In Eq. (57), the index n can take the values $n = k, k+1, \ldots, q_{\mathrm{cut}}$. The cutoff q_{cut} of the network's degree distribution $P(q)$ depends on details of a specific network and its size N.

Although we derived Eq. (57) by using heuristic arguments, this equation is exact for uncorrelated random graphs in the limit $N \to \infty$. Equation (57) may be strictly derived by differentiating Eq. (5) over p and using Eq. (24).

The set of Eqs. (57) with n from k to q_{cut} is a complete set of nonlinear equations which determine $M_k(n)$ as a function of p. The nonlinearity is due to the functional dependence of r on $M_k(n)$.

Summing over n from k to q_{cut} on the left and right hand sides of Eq. (57), we obtain Eq. (3). If we know $M_k(n)$ for an initial network, i.e., at $p = 1$, then we can use Eq. (57) and find the evolution of $M_k(n,p)$ with decreasing p. Inserting Eqs. (11) and (12) into (13), we can determine the order parameter R as a function of p,

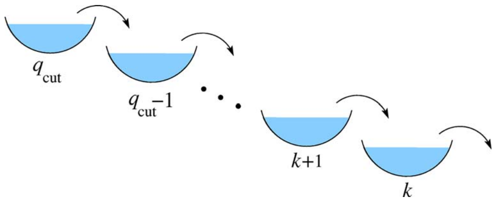

FIG. 8. (Color online) Schematic picture of a relaxation process described by Eq. (60). An initial distribution $M_k(n,t=0)$ over states with $n=q_{\rm cut},q_{\rm cut}-1,\ldots,k$ relaxes into the final state $\{M_k(n)=0\}$ due to transitions of vertices from a state with degree $n+1$ to a state with degree n. Here $q_{\rm cut}$ is the maximum degree.

$$R = 1 - \left[z_1^{-1} \sum_{n=k}^{q_{\rm cut}} n M_k(n,p) \right]^{1/2}. \tag{59}$$

Alternatively, we could find the order parameter $R(p)$, solving Eq. (7), and afterwards obtain $M_k(n)$ from Eq. (5).

X. MAPPING TO A COOPERATIVE RELAXATION MODEL

Let us consider the *k*-core percolation as an evolutionary process. At time $t=0$ we have an initial uncorrelated network with the *k*-core. During a time interval Δt we remove at random a fraction $\Delta p/p=\Delta t$ of occupied vertices from the network. This means that the occupation probability p decreases in time as $p=e^{-t}$. With this substitution, Eq. (57) takes the form

$$\frac{dM_k(n,t)}{dt} = - M_k(n,t) - rnM_k(n,t) + r(n+1)M_k(n+1,t). \tag{60}$$

This rate equation describes the relaxation of an initial distribution $\{M_k(n,t=0)\}$ to the final state with the destroyed *k*-core, i.e., $\{M_k(n)=0\}$, due to the chain of transitions of vertices from states of degree $n+1$ to states of degree lower by one: $n+1\to n$, see Fig. 8. Note that we consider only relaxation in states with $n\geqslant k$, assuming that vertices of degree less than k are pruned instantly [31–33]. The parameter r plays the role of the characteristic scale of the relaxation rate. This relaxation is a cooperative process due to the functional dependence of r on $M_k(n,t)$, see Eq. (58). At time $t_c(k)=\ln[1/p_c(k)]$ this model undergoes a dynamical phase transition. Using Eq. (4), we find that the total number $M_k(t)$ of vertices in the *k*-core has a singular time dependence near $t_c(k)$:

$$M_k(t) - M_k[t_c(k)] \propto [p(t) - p_c(k)]^{\nu} \propto [t_c(k) - t]^{\nu}. \tag{61}$$

The critical exponent $\nu=1/2$ is valid for $k\geqslant 3$. Inserting Eq. (48) into Eq. (58), we find that the relaxation rate diverges at the critical time $t_c(k)$,

$$r \propto [p(t) - p_c(k)]^{-1/2} \propto [t_c(k) - t]^{-1/2}. \tag{62}$$

Note that in accordance with the results obtained in Sec. II, the characteristic scale r of the relaxation rate also determines the mean size of the region in the *k*-core cropped out due to the deletion of a vertex. In its turn, r is approximately equal to the correlation length λ, i.e., the larger is the correlation length the larger is the relaxation rate. This is in contrast to the usual critical slowing down of the order parameter relaxation for continuous phase transitions. In the latter case, the larger is the correlation length the smaller is the relaxation rate $r\approx\lambda^{-z}$, where z is a dynamical critical exponent.

XI. CONCLUSIONS

In this paper we have explained the nature of the *k*-core percolation transition in uncorrelated networks. To obtain our results, we used heuristic arguments and developed an exact theory. Let us list the main features of the quite unusual *k*-core percolation transition: (i) a jump emergence of the *k*-core, (ii) the critical singularity in the phase with the *k*-core, (iii) the absence of any critical effects — strong correlations, divergent “susceptibilities,” etc. — on the “normal phase” side. We had to reveal the meaning of the order parameter in this problem, to explain the nature of the jump, to find the origin of the singularity, to indicate a “physical” quantity, which diverges at the critical point, to indicate long-range correlations, specific for this transition.

We have shown that the order parameter in this problem is simply expressed in terms of the relative number of edges in the *k*-core, see relation (16). The tree ansatz has allowed us to find the *k*-core order parameter and other *k*-core characteristics of various uncorrelated networks.

We have found that the unique properties of the *k*-core percolation transition are essentially determined by the corona subset of the *k*-core, that is, by vertices with exactly *k* connections to the *k*-core. These are the “weakest” vertices in the *k*-core. The critical correlations in the *k*-core are due to the correlations in the system of the corona clusters.

In the “*k*-core phase,” the corona clusters are finite, but their sizes and long-range correlations grow as the network approaches the *k*-core percolation threshold. The mean size of a corona cluster to which a randomly chosen corona vertex belongs diverges at the *k*-core percolation threshold. This quantity plays the role of a critical susceptibility in this problem. So, the *k*-core percolation threshold coincides with the percolation threshold for corona clusters, and the *k*-core phase is the “normal” phase for the corona. The dramatic difference from the ordinary percolation is that the corona disappears on the other side of the threshold, and so critical fluctuations in the phase without the *k*-core are absent.

For understanding the nature of this transition, we have studied the process of the destruction of the *k*-core due to the random deletion of vertices. The deletion of a vertex in the *k*-core results in the clipping out the entire adjacent corona clusters from the *k*-core due to the domino principle. This effect is enormously increased when corona clusters become large — near the *k*-core percolation threshold. In the threshold, the removal of a tiny fraction of vertices results in the complete collapse of the corona and the *k*-core. In this respect, the *k*-core percolation problem can be mapped to a model of cooperative relaxation, which undergoes critical relaxation with a divergent rate at some critical moment.

To conclude, let us indicate a possible application — a social network model, where social links connect individuals. Each of vertices (individuals) in our model may occur in one of a few states — distinct beliefs, opinions, religions, ideologies, diseases, etc. We assume that each vertex takes a specific state if at least k its neighbors are in this state. Is it possible that in this social net a giant, say, religious group will emerge? The answer is yes if the network has the giant k-core. A giant community of individuals being in the same state forms the k-core of this network. We believe that our results are applicable to a variety of complex cooperative systems of this kind.

ACKNOWLEDGMENTS

This work was partially supported by POCTI Projects Nos. FAT/46241, MAT/46176, FIS/61665, and BIA-BCM/62662, and DYSONET. Authors thank J.G. Oliveira for help in numerical calculations.

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[26] Note that vertex i is not included in the number $N_{\rm crn}$ even if this vertex belongs to the corona.
[27] This important statement can be proved by the direct calculation of the conventional probability $P_k(n'|n)$ that if a vertex in the k-core is of degree n, then its nearest neighbor in the k-core subgraph is of degree n'. This probability turns out to be independent of n.
[28] Note that in this formulation, the value of the order parameter [let us denote it by $\widetilde{R}(p)$] turns out to be different from $R(p)$ defined in Sec. III. The reason is that in the definition of $R(p)$ for a damaged network, the edges of the original undamaged net were used [see, e.g., Fig. 22(a)]. So in the damaged network, these edges may be absent. These two order parameter values are simply related to each other: $1-R(p)=\sqrt{L(p)/L}[1-\widetilde{R}(p)]$, where $L(p)$ is the total number of edges in the damaged net. This trivial renormalization directly follows from relation (16).
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